NEW RESULTS ON THE IDENTIFICATION OF NORMAL MODES FROM EXPERIMENTAL COMPLEX MODES

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Complex modes are associated to first-order representations of structural dynamics and normal (real) modes to undamped second-order models. As normal modes are needed for comparison with undamped finite element models, their determination is a key issue. Damping introduces coupling between the normal modes so that, in general, there is no simple relation between normal and complex modes. The complex modes of a second-order model are said to be complete. It is shown that a set of complex modes is complete if it verifies a defined properness condition which is used to find complete approximations of identified complex modes. The possibility of finding a complete approximation of a set of complex modes is linked to damping properties. It is shown that the traditional assumption of proportional damping can be extended to groups of vectors and that the associated separation criterion enables the selection of groups of complex modes for which a complete approximation can be found. Co-ordinate systems used to represent the mass, damping and stiffness properties determined by the proposed approach are also discussed. An experimental application of the proposed methodology is made using six independent tests of the MIT/SERC interferometer testbed. The results clearly demonstrate applicability to real structures with high modal densities and significant non-proportional damping.

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1. INTRODUCTION

Analytical models of structures are generally undamped. Therefore, test models used for validation and updating must characterise separately mass, damping and stiffness properties. Most identification procedures provide models of the measured input/output response which are first-order and thus have complex modes. The assumption of proportional damping has often been used to link the measured complex modes to the normal modes (which characterise the mass and stiffness properties) and the modal damping matrix (which contains all the damping information), but only very simple structures are proportionally damped. The error linked to the use of the proportional damping assumption is, however, negligible for very low levels of damping and significant spacing of modal frequencies [1, 2].

Modern structures tend to have both higher modal densities and higher levels of passive damping in order to reduce noise and vibration. The conditions which traditionally allowed the creation of accurate models using the proportional damping assumption are thus rarely met, and the use of a non-proportional damping model is now, in a majority of cases, a necessity for high accuracy modeling.

The identification of non-proportionally damped normal mode models has been and is still the object of a number of studies. Reviews of the topic can be found in [3–5]. Two
main approaches are considered. Some methods directly identify models in a second-order form [6–11]. These models can then be used to determine normal modes and in certain cases a non-proportional damping matrix. Other methods assume that a set of complex modes have been identified from test data and use constraints imposed by the orthogonality conditions to define a transformation to a normal mode model [10–14]. The present paper discusses a method of the second category, and demonstrates its effectiveness using test data taken on the MIT/SERC interferometer testbed.

Section 2 reviews concepts linked to complex and normal modes. Definitions of complex modes of a second-order model (such complex modes are called complete [4]) are recalled. It is shown that completeness is equivalent to a condition of properness, which will be used in the proposed identification procedure. Definitions of normal modes and of the proportional damping assumption are reviewed. A new assumption of proportional damping by bloc is introduced to show that completeness should be applied to groups of vectors. A criterion to select such groups is proposed. Finally, the arbitrariness in the choice of model degrees of freedom (dof), and associated model mass, damping and stiffness matrices, is addressed.

In Section 3, a procedure is proposed for the determination, from identified complex modes, of second-order models with mass, damping and stiffness properties. The key steps are the scaling of complex mode output matrices, the choice of co-ordinate system, the approximation by a proper complex mode model and the transformation to the second-order model form.

In Section 4, the efficiency of the overall approach is shown for the experimental case history of the MIT/SERC interferometer testbed. In particular, the ability to determine accurately non-diagonal coupling terms in the damping matrix is demonstrated.

2. DEFINITIONS AND CONCEPTS

One considers linear structures represented by models of the second-order form

\[
[Ms^2 + Cs + K]N \times N \{q\}N \times 1 = [b]N \times N_A \{u(s)\}N_A \times 1
\]

\[
[y(s)]N_S \times 1 = [c]N_S \times N \{q\}N \times 1
\] (1)

In such models, the response is fully described by a finite number of dof \(q\) that depend on time/frequency. The dynamic stiffness matrix \(Ms^2 + Cs + K\) gives the relation between the response of the model dof \(q\) and the model loads \([b]\{u(s)\}\). The input \(b\) and output \(c\) shape matrices are discussed in Section 2.3.

2.1. COMPLEX MODES OF A SECOND-ORDER MODEL

Complex modes of a second-order model are defined as solutions of the generalised eigenvalue problem (which corresponds to a first-order representation of model (1) [4, 5])

\[
\begin{bmatrix}
  C & M \\
  M & 0 \\
\end{bmatrix} \theta A + \begin{bmatrix}
  K & 0 \\
  0 & -M \\
\end{bmatrix} \theta = 0
\] (2)

The \(2N\) complex eigenvectors \(\theta_j\) and eigenvalues (poles) \(\lambda_j = -\zeta_j \omega_j \pm i \omega_j \sqrt{1 - \zeta_j^2}\) come
in complex conjugate pairs. The eigenvectors \( \theta_j \) can be decomposed into a displacement vector \( \psi_j \) and a velocity vector \( \psi_j^* \).

\[
[\theta]_{2N \times 2N} = \begin{bmatrix} \psi_j \end{bmatrix}_{2N \times 2N} \text{ and } A = \begin{bmatrix} \lambda_j & \cdot \\ \cdot & \cdot \end{bmatrix}_{2N \times 2N}
\]

In an abuse of language, the pair displacement vector \( \psi_j \) and pole location \( \lambda_j \) will also be called a complex mode (there is no loss of information since this mode fully characterises the complex eigenvector \( \theta_j \)).

A set of \( N \) pairs of conjugate complex modes is called complete [4] if it contains all the modes of a second-order model with \( N \) dof. If \( NS \) (the number of sensors) is larger than \( N \) (the number of pairs of complex modes), the measurement of these complex modes can still be written as the product \([c]_{NS \times N}^T[\psi]_{N \times 2N}\) (where the output shape matrix \( c \) is real). This and the properness condition discussed below are characteristics of complete sets of complex modes.

Two orthonormality conditions are necessary and sufficient for a set of \( 2N \) vectors \( \psi \) and poles \( A \) to be the complete set of complex modes of a model with \( N \) dof:

\[
\theta^T \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \theta = \psi^T C \psi + A \psi^T M \psi + \psi^T M \psi A = \nu
\]

\[
\theta^T \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix} \theta = \psi^T K \psi - A \psi^T M \psi A = -\nu A
\]

where the modal scaling coefficients (terms of the diagonal matrix \( \nu \)) are arbitrary complex numbers. Two scaling conditions (choices of \( \nu \)) are traditionally considered in the literature.

\( \nu = A - A^* = 2i \text{ Im}(\lambda_j) \) considered in [10, 12], has the advantage of an easy link to the mass normalisation of normal modes. As discussed in more detail in Section 2.2, a proportionally damped mode is such that all the components of the complex mode \( \psi_j \) have the same phase. In fact, with the considered normalisation, the complex mode is real valued and equal to the normal mode \( \phi_j \). Section 2.2 discusses when non-“real” complex modes (with components having different phases) can be related to a second-order model that has normal modes (which are real by definition).

\( \nu = I \) is considered here and in [3, 14–17]. With this normalisation, complex modes of a proportionally damped system are on a \(-45^\circ\) line (rather than being real valued). Simple algebraic manipulations of the orthogonality conditions (4–5) lead to two equivalent expressions

\[
\begin{bmatrix} C & M \\ M & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & M^{-1} \\ M^{-1} & -M^{-1}CM^{-1} \end{bmatrix} = \theta \theta^T = \begin{bmatrix} \psi \psi^T \\ \psi A \psi^T \\ \psi A \psi^T \end{bmatrix}
\]

\[
\begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix}^{-1} = \begin{bmatrix} K^{-1} & 0 \\ 0 & -M^{-1} \end{bmatrix} = -A^{-1} \theta \theta^T = -\begin{bmatrix} \psi A^{-1} \psi^T \\ \psi \psi^T \\ \psi A \psi^T \end{bmatrix}
\]
which are verified if, and only if, the model matrices can be expressed as follows
\[ M = (\psi A \psi^T)^{-1}, \quad C = -M\psi A \psi^T M, \quad \text{and} \quad K = -(\psi A^{-1} \psi^T)^{-1} \] (8)
and the complex modes verify the condition
\[ [\psi][\psi]^T = 0. \] (9)

It can also be easily shown using the orthogonality conditions that the response of model (1) is exactly given by
\[ \{y\} = [c][Ms^2 + Cs + K]^{-1}[u] = \sum_{j=1}^{\infty} \frac{(c\psi_j)(\psi_j^T b)}{s - \lambda_j} \{u\} \] (10)

At high frequency, mass contributions of a second-order model become predominant so that transfer functions have an asymptote in \(1/\omega^2\). Thus the velocity contribution \(sy: 0\) as \(s: a\). By applying this condition to the complex mode form (10) of the model, one finds that \(\sum_{j=1}^{\infty} (c\psi_j)(\psi_j^T b) = 0\) for an arbitrary \(b\) and \(c\). This is clearly equivalent to the condition (9). Systems that have no high frequency contribution are called proper. The condition (9) will thus be called the properness condition (different forms of this condition can be found in [10, 12]). The properness condition means that the considered complex modes are associated by equation (8) to a second-order model of the form (1). Completeness and properness are thus equivalent. A practical use of the properness form (9) will be made in Section 3.

2.2. NORMAL MODES AND ASSUMPTIONS ON DAMPING

Normal modes are defined as eigenvalues \(\omega_j^2\) and eigenvectors \(\{\phi_j\}\) of the conservative (based on mass and stiffness properties only) eigenvalue problem
\[ (-[M]\omega_j^2 + [K])\{\phi_j\} = 0 \] (11)
The scaling of the normal modes shapes \(\phi\) can be chosen (unity generalised mass) so that
\[ [\phi]^T[M][\phi]_{N \times N} = [I]_{N \times N} \quad \text{and} \quad [\phi]^T[K][\phi] = [\Omega_j^2] \] (12)
where \(\Omega_j^2\) is the diagonal matrix of normal mode frequencies squared.

Using the normal modes as basis functions to describe the motion of the system leads to a normal mode model of the damped system
\[ [I]s^2 + [\Gamma][s + [\Omega_j^2]]\{p\} = [\phi^T b][u], \quad \{y\} = [c\phi][p] \] (13)
which uses the normal mode or principal co-ordinates \(p\) where the (modal) mass matrix is \(I\), the damping \(\Gamma = \phi^T C \phi\) and the stiffness \(\Omega_j^2\).

In many cases, one considers a partition of equations (13) in two sets of modes \(p_1\) and \(p_2\):
\[
\begin{bmatrix}
I & \Gamma_{11} & \Gamma_{12} \\
\Gamma_{21} & \Gamma_{22} & \Omega_j^{(1)} \\
\end{bmatrix} \begin{bmatrix}
p_1 \\
p_2 \\
\end{bmatrix} = \begin{bmatrix}
\phi_1^T b \\
\phi_2^T b \\
\end{bmatrix}, \quad y = [c\phi_1 \ c\phi_2] \begin{bmatrix}
p_1 \\
p_2 \\
\end{bmatrix} \] (14)
In the form (14), the equations in \(p_1\) and \(p_2\) are only coupled by the off-diagonal damping terms \(\Gamma_{12} = \Gamma_{21}^{(1)}\).
It is often assumed that the modal damping matrix $G$ is diagonal. The response of each normal mode is then completely uncoupled from the response of other modes. This mathematically motivated assumption, called proportional or modal damping, has been well understood and widely used since the original contribution of Caughey [18]. Recently Liang et al. [19] introduced the idea that the assumption of proportional damping should be applied to a single mode. Thus, a proportionally damped mode corresponds to $p_i$ with a single normal mode and $G_{12} = 0$. Such a normal mode is clearly associated to a pair of complex modes which, for $v = I$ scaling, is given by

$$\omega_i = \omega_{NJ}, \quad \zeta_i = \Gamma_{ij}/2\omega_i \quad \text{and} \quad \psi_i = \phi_i/\sqrt{2i\omega_i\sqrt{1 - \zeta_i}}$$

(15)

For a non-proportionally damped, mode $\omega_i \neq \omega_{NJ}$. However, for light damping, the difference is usually small.

Although there is no reason for any structure to be truly proportionally damped, the errors introduced by ignoring the coupling terms are often negligible [1, 2]. Hasselman [2] proposed a condition characterising the decoupling of two modes,

$$2\zeta_i|\omega_i - \omega_k| \ll 1$$

(16)

which corresponds to the fact that the coupling introduced by non-proportional damping on an initially proportionally damped model can only be significant if the cross-modal impedance is low [i.e. when (16) is large]. Note that the off-diagonal coupling term is fundamentally limited by the diagonal damping terms ($\gamma_k \leq \sqrt{\gamma_i\gamma_j}$ since the damping matrix is known from physics to be positive-definite), so that the criterion (16) can be based on the diagonal terms only without the restriction in [2].

In most structures, some modes are close to each other and, even for small damping levels, do not verify the separation criterion (16). It is thus useful to introduce the assumption of proportional damping by bloc as the fact that a group of modes $p_i$ is decoupled from all other modes (i.e. $\Gamma_{ij} = 0$). For the same reasons that make it a valid indicator for proportional damping models, the separation criterion (16) gives a way to select groups of modes (the criterion is applied for modes of different groups but not within each group). A more detailed discussion of the assumption of proportional damping by bloc can be found in [20].

The decoupling of equations in (14) implies that the bloc of $NR$ normal modes is associated to $2NR$ complex modes ($NR$ pairs of conjugates). Furthermore these complex modes can be exactly projected on the basis of the $NR$ normal modes (they form a complete set). The assumption of proportional damping by bloc thus states that under certain conditions [and the separation criterion (16) is a practical indicator] there exists a good complete approximation of a group of complex modes.

2.3. DEGREES OF FREEDOM AND PHYSICAL PROPERTIES

The system can be excited by external forces described by a vector of time/frequency-dependent coefficients $u$ (called inputs). One assumes that the model loads can be linearly related to the input coefficients $[b]_i u(s)$. By reference to control theory, the matrix $b$ will be called an input shape matrix. Physical displacements $y$ (translations, rotations, strains, stresses, etc.) are assumed to be linearly related to the model dof $\{y(s)\} = [c]_i q$ (where $c$ will be called an output shape matrix).

For reciprocal systems, the model can be constructed so that mass $M$, damping $C$ and stiffness $K$ are symmetric matrices. The mass matrix is also positive definite (since kinetic energy is always positive), the damping matrix positive semi-definite (for a passive system), and the stiffness matrix positive semi-definite (rigid body modes of the stiffness matrix span the kernel of the stiffness matrix). For colocated inputs and outputs (i.e. such that $y\dot{u}$ is
a power input), the input and output matrices are the transpose of each other \( c_c = b_c^T \) and this is independent of the co-ordinate system \( q \) chosen.

Properties that are theoretically independent of the way they are measured or estimated will be called physical properties. In a modal test, one measures the relation between inputs \( u \) and outputs \( y \). This relation is (at least theoretically) invariant for any model transformation.

Degrees of freedom are not inherently physical quantities. More precisely, a non-singular transformation \( \{ q \} = [T] \{ \hat{q} \} \) of the dof does not modify the model (relations between \( u \) and \( y \)). Usual choices that lead to physical dof are sensor dof (one seeks a reciprocal model where the \( c \) matrix is the identity) or principal dof (one seeks a reciprocal model where the mass matrix is the identity and the stiffness matrix is diagonal).

The modal input, output shape matrices \((c\phi, \phi^T b \text{ and } c\psi, \psi^T b)\) which characterise how modes are seen/excited by certain sensors/actuators are physical quantities. The normal \( \phi \) and complex \( \psi \) modes are only physical with respect to a physically defined choice of co-ordinates. Finally this co-ordinate choice is fully characterised by the choice of \( c \) \((b\) being defined by reciprocity).

3. TRANSFORMATIONS OF EXPERIMENTAL MODELS

The response of a linear and diagonalisable model can always be written as a sum of the form (10). In experiments, however, only modes within the measured frequency range have a significant influence on the response and can thus be measured with accuracy. Other modes have contributions which can only be approximated. The contribution of these modes \([E(s)]\) will not be used for the creation of a second-order model. It is thus assumed that an initial model has been identified and can be written in the form

\[
[H(s)] = \sum_{\text{dynamic} \text{ modes in the BW}} \left\{ \frac{[R_j]}{s - \lambda_j} + \frac{[R_j]^\ast}{s - \lambda_j^\ast} \right\} + [E(s)]
\]

with the residue matrix theoretically given by \([R_j]_{NS \times NA} = \{c\psi\}_{NS \times 1} \{\psi^T b\}_{1 \times NA}\).

The notions mass, damping and stiffness properties have only been defined for a model of the form (1) which represents a dynamic flexibility (transfer from force inputs \( u \) to displacement outputs \( y \)). The transfer functions \( H \) are thus assumed to represent a dynamic flexibility. If velocity or acceleration is measured, the exact frequency domain integration should be used to obtain displacement. Note that it is not always easy to ensure that a dynamic flexibility is measured.

The number of retained pairs of complex modes will be noted \( NR \). As detailed in Section 2.2, one may seek a second-order representation for any group of modes that verify the separation criterion (16). The following procedure can thus be applied once to a group containing all the complex modes identified or sequentially to different non-overlapping groups selected with (16).

Although the results of Section 4 are based on the IDRC algorithm [15, 16, 21] which directly identifies a model of the form (17), the results of most identification algorithms can be transformed to this form and thus be used here.

The procedure detailed below can be decomposed into the following steps:
(1) determination of scaled complex mode output shape matrices \((c\psi)_{ID}\) (Section 3.1);
(2) choice of $c_{id}$ and determination of the associated $\psi_{id}$ (Section 3.2);
(3) determination of $\bar{\psi}_{id}$ (a proper approximation of $\psi_{id}$) (Section 3.3);
(4) computation of mass, damping and stiffness matrices using $\bar{\psi}_{id}$. $A_{id}$ in (8), the resulting model is given by (where $b_{id}$ is linked to $c_{id}$ by reciprocity)

$$[[M_{id}]s^2 + [C_{id}]s + [K_{id}]]\{q_{id}\} = [b_{id}]_{NR \times N_A}\{u\},$$

$$\{y\}_{NS \times 1} = [c_{id}]_{NR \times 1} + [E(s)]_{NS \times N_A}\{u\} \quad (18)$$

(5) determination of the normal modes $\phi_{id}$ using (11) on the $M_{id}$, $K_{id}$ matrices;
(6) construction of the normal mode model by projection

$$[[I]s^2 + [G_{id}]s + [\Omega_{id}]]\{p\} = [\phi_{id}^T b_{id}]\{u\}, \quad \{y\} = [c_{id} \phi_{id}]\{p\} + [E(s)]\{u\} \quad (19)$$

3.1. INITIAL STEP: COMPLEX MODE INPUT AND OUTPUT SHAPE MATRICES

The IDRC algorithm and some other methods do not take into account the fact that the residue matrix $R_i$ is the product of the modal output $\{c\psi_i\}$ (characterising how the mode is seen at different sensors) by the modal input $\{b\psi_i\}^T$ (characterising how the mode is excited by different actuators). Since $\{c\psi_i\}$ is a column vector and $\{b\psi_i\}^T$ a row vector, their product, called a dyad, is of rank 1.

The IDR algorithm [16] provides a way to take this constraint into account. The idea is to find a rank 1 approximation $\bar{R}_i$ of the identified matrix $R_i$. To do this, one uses the singular value decomposition $R_i = \Sigma U_i \sigma_i V_i^T$ of the identified matrix and takes

$$[\bar{R}_i] = \{U_i\} \sigma_i \{V_i\}^T \quad (20)$$

By definition of the singular value decomposition, this choice minimises the matrix norm of the difference $R_i - \bar{R}_i$. The matrix norm of the difference is in fact given by $\sigma_2$, so that a measure of the quality of this approximation is given by the ratio $\sigma_2/\sigma_1$ that should be much less than 1.

For multiple modes (axisymmetric structures for example), residue matrix should have the same rank as the multiplicity. This can be obtained by keeping different dyads $\bar{R}_j = U_j \sigma_j V_j^T$, $\bar{R}_i = U_i \sigma_i V_i^T$, etc. which give the contributions of the different orthogonal modeshapes of this multiple pole (more details in [15, 16]).

For a rank 1 residue matrix which can always be written in the form (20), $U_i$ and $V_i^T$ are respectively proportional to estimates of $c\psi_i$ and $\psi_j^T b$. But without further information one cannot, from the equality $U_i \sigma_i V_i^T = c\psi_i \psi_j^T b$, determine the proper scaling coefficients to determine uniquely the normalised estimates of $c\psi_i$ and $\psi_j^T b$ (multiplying $c\psi_i$ by $z$, and $\psi_j^T b$ by $1/z$ leaves the equation unchanged).

For a collocated transfer function (e.g. transfer function from force input to displacement at the same location and in the same direction) of a reciprocal system, the modal inputs and outputs are equal (because of reciprocity): $c\psi_i = \psi_j^T b$, for all modes $j$ at the considered location $i$ (this is the duality property introduced in Section 2.3). This second set of equations allows the experimental determination of scaled input and output matrices

$$(c\psi)_{id} = \sqrt{\frac{U_{id} \sigma_{id} V_{id}^T}{U_{id}}} U_i \quad \text{and} \quad (\psi_j^T b)_{id} = \sqrt{\frac{U_{id} \sigma_{id} V_{id}}{V_{id}^T}} V_i^T \quad (21)$$

If more than one transfer function is collocated the condition is redundant and must be imposed using some error minimisation scheme which should diminish the overall sensitivity to errors on the collocated residues.
The reciprocity condition gives a relation between input and output shape matrices. It is thus possible to consider output shape matrices only and not lose information. This will be done in all following sections.

3.2. CHOOSING DEGREES OF FREEDOM

The present section discusses how, for given complex mode output shape matrices \((\psi_{ID})\), the real-valued output \([c_{ID}]_{NS \times NR}\) shape matrix \((b_{ID})\) is defined using reciprocity and associated complex modeshapes \([\psi_{ID}]_{NR \times NR}\) should be chosen. It is always assumed that there are more sensors than modes \((NR \leq NS)\).

If the number of sensors is equal to the number of normal modes, the choice of a \(c_{ID}\) and \(\psi_{ID}\) involves no constraint: for an arbitrary non-singular matrix \(c_{ID}\) (e.g. \(c_{ID} = I\)) the modeshapes are defined by \(\psi_{ID} = c_{ID}^T(\psi_{ID})\). In other cases, one must define \(c_{ID}\) and \(\psi_{ID}\) so as to optimise the overall match \((c_{ID}^T\psi_{ID} versus \{c_{ID}\}_{ID})\).

When using the complex mode scaling \(n = L - L^* = 2i \text{Im}(\lambda_i)\), the modal output of a proportionally damped system is real. Assuming that the complex modes should be mostly “real” even for non-proportionally damped systems, it was proposed in [10, 12] to use the real part of the complex mode. For the unity scaling \(n = I\) used in this paper, this choice becomes

\[
c_{ID} = \text{Re}((c\psi_{ID})\sqrt{i}/\sqrt{i})
\]  

Another approach was implemented in [21] and can be derived as follows. With a good model, one should have from the inversion expression \((9)\)

\[
cM^{-1}c^T \approx (c\psi_{ID}A_{ID}(c\psi_{ID})^T
\]

Furthermore, if the \(NR\) complex modes kept verified the assumption of proportional damping by bloc, the matrix \((c\psi_{ID}A_{ID}(c\psi_{ID})^T)\) would be positive definite of rank \(NR\). In practice one can verify that this matrix has \(N\) positive eigenvalues and choose the \(N\) corresponding eigenvectors for \(c_{ID}\).

For a chosen output shape matrix \(c_{ID}\), an estimate \(\psi_{ID}\) of the complex modeshapes can be found as a solution of a least-squares problem

\[
\psi_{ID} = \arg\min_{\psi_{ID}} (\text{trace}((c\psi_{ID} - c_{ID}\psi_{ID})^T(c\psi_{ID} - c_{ID}\psi_{ID})))
\]

Theoretically, the choice of \(c_{ID}\) is irrelevant since a change of co-ordinates (transformation \(\{q\} = [T]\tilde{q}\)) can be performed at any point. In practice, however, the proposed transformation to a second-order form induces an approximation of the modeshapes \(\psi_{ID}\) and the choice of co-ordinates is important. In some cases, the choice linked to \((23)\) gave better result than of \((22)\). Further investigation is still needed however.

3.3. MASS, DAMPING, STIFFNESS AND NORMAL MODES

One has now selected dof (chosen \(c_{ID}\)) and determined associated complex modes \(\psi_{ID}\). This section discusses the determination of mass, damping and stiffness properties in this co-ordinate system and the transformation to a normal mode model.

The use of the inversion formulae is not new \([10, 12, 14]\) but it has never been proposed to enforce the properness condition \((9)\) on the identified complex modes \(\psi_{ID}\) as a first step.

In a fashion similar to the IDRM algorithm discussed in Section 3.1, one thus seeks a proper approximation \(\tilde{\psi}_{ID}\) of the identified modes \(\psi_{ID}\). This approximation is found as a solution of the minimisation problem (minimise difference while satisfying the properness condition):
It can be shown [15] that a closed form solution of this constrained optimisation problem is

$$\psi_{ID} = (I - \delta^* \delta)^{-1}(\psi_{ID} - \delta \psi_{ID})$$

(26)

where \(\delta\) is solution of the following algebraic Riccati equation

$$[\psi_{ID} \psi_{ID}^T]_{NR \times NR} - [\delta \delta^T]_{NR \times NR} - [\psi_{ID} \psi_{ID}^T][\delta \delta^T]_{NR \times NR} = 0,$$

(27)

which can be solved numerically as done in the implementation (21) of the methods presented here.

The use of the transformations (8) with \(\psi_{ID}\) results in mass, damping and stiffness matrices linked to the retained modes [model (18)]. By solving the associated conservative eigenvalue problem (11), one can find estimates of the normal modes (in the co-ordinate system associated to the chosen \(c_{ID}\)) or the normal mode output shape matrices which are physical (independent of the choice of a co-ordinate system). In the resulting normal mode model (19), the modal damping \(G_{ID} = \phi_{ID}^T C_{ID} \phi_{ID}\) and modal stiffness \(\Omega_{ID} = \phi_{ID}^T K_{ID} \phi_{ID}\) are also physical quantities.

4. EXPERIMENTAL DEMONSTRATION

To highlight the efficiency of proposed methodology for the identification of damped normal mode models, applications to the case of the MIT/SERC interferometer testbed will be detailed here (see [22] for a description of the testbed and its objectives). As shown in Fig. 1, the testbed is composed of six triangular truss beams forming a tetrahedron measuring 3.5 m on each side, representing a scaled model of a space-based interferometer.

![Figure 1. Configuration of the MIT/SERC interferometer testbed modal test. •, accelerometer locations; ▲, numbered shaker locations.](image-url)
To create experimental models, a modal test of the interferometer testbed was performed, using an external shaker at six different locations (▲, Fig. 1), and 28 accelerometer measurements (●, Fig. 1; some locations having measurements in multiple directions). The experimental models considered in this paper are derived from the measured transfer functions for single shaker locations (a single shaker was moved to six positions). The six independent models obtained were compared to demonstrate the quality of predictions.

4.1. IDENTIFICATION OF COMPLEX MODES

For each of the six configurations, a pole/residue model of the form (17) was identified using the IDRC algorithm [15, 16, 21]. Figure 2 shows one of the identified transfer functions and the associated model. Difficulties for the identification of this structure are the local mode at 44.2 Hz (this mode has a small contribution on all but a few sensors) and the two very close modes (36.2 and 36.5 Hz with damping ratio of 3.1 and 0.6%, respectively). The vertical dotted lines in the figure indicate the imaginary part of the poles.

The first step of the procedure detailed in Section 3 is the determination (21) of output shape matrices ($c_{\psi}$) scaled using the reciprocity condition. Figure 3 shows that for mode 7 at 36.5 Hz, the results were extremely consistent for the six considered models (dotted lines link outputs of corresponding sensors).

The models identified here match accurately the data which has very little noise. Such accuracy can clearly not always be achieved. Robustness of the following steps of the proposed method to identification errors is thus an important issue which was not addressed.

4.2. IDENTIFICATION OF MASS DAMPING AND STIFFNESS PROPERTIES

The results shown in this section correspond to the application of the proposed method to the group of the first nine modes (including the local mode at 44.2 Hz). Good results

Figure 2. Frequency response function of accelerometer 18 actuator location 2. ——, Measured FRF; ---, model (IDRC [21] result).
would also be found if the method was applied on smaller groups (modes 1–4, modes 5–8, mode 9, etc.) which verify the separation criterion (16).

For comparisons, it is possible to compute the complex modes of the identified model (19) and to reconstruct the associated complex mode output shape matrix (exactly equal to $c_{ID} c_{flp0ID}$). For a proportionally damped mode $j$ ($\Gamma_{ID}$ forms a block for this mode), the complex mode would, based on (15), be equal to $(c_{ID} \phi_{ID})/\sqrt{2i\omega_j \sqrt{1 - \zeta_j^2}}$. Figure 4 shows a comparison for mode 6 (at 36.2 Hz). The complex mode identification with application of reciprocity leads to $(c_{ID} \phi_{ID})$ (non-proper model, shown as +) and one notes that modal outputs exhibit a significant phase spread. The complex mode output $c_{ID} \phi_{ID}$ (proper complex mode model exactly associated to a non-proportionally damped second-order model, shown as $\bigcirc$) shows a minor difference with the initial complex mode whereas

![Figure 3](image1.png)

**Figure 3.** Coherence between the six tests of the complex mode output matrix $(c\psi)_{ID}$ for mode 7 (estimates at the same sensor are linked by dotted lines).

![Figure 4](image2.png)

**Figure 4.** Comparison of complex modal outputs of mode 6: +, $(c\psi)_{ID}$; $\bigcirc$, $c_{ID} \phi_{ID}$; $\times$, $(c_{ID} \phi_{ID})/\sqrt{2i\omega_j \sqrt{1 - \zeta_j^2}}$. Dotted lines link results of corresponding sensors.
(c_{ijd}\phi_{ijd})/\sqrt{2\left|\omega_i\right|\sqrt{1-\xi_j^2}} (the proportionally damped model found by setting off diagonal terms in \(\Gamma_{ijd}\) to zero, shown as \(\times\)) gives a very poor approximation of the initial identification result.

Measured and predicted frequency response functions allow good comparisons over a certain frequency range, but only for a single location. Figure 5 shows for actuator 5 sensor 3, that the proposed method (model (19)) gives very good results whereas a proportionally damped model [model (19) with off-diagonal terms of \(\Gamma_{ijd}\) set to zero] would be inaccurate both in magnitude and phase (more than 10dB in magnitude and 30° in phase). This huge difference is linked to the fact that modes 6 and 7 are very coupled and is not found for all transfer functions.

The six tests performed here give independent estimations of model (19) [except for the \(E(s)\) term which is expected to differ from test to test]. The consistency of normal mode shapes \(c_{i\omega}\phi_{ijd}\) between the six tests is extremely good. Modal assurance criterion values are above 0.99, apart from a few modes that are not excited well enough to be correctly identified (e.g. mode 9 is mostly localised to leg IV and is not well identified for the shaker positioned on leg I).

In terms of input/output matrix scaling (generalised mass) the results are not as good but still excellent by industry standards (well below 10% for the majority of cases). The availability of different estimates of the scaled I/O matrices permitted useful comparisons, which allowed the elimination \textit{a posteriori} of modes that could be diagnosed as not well identified. For other tests, the use of multiple excitation patterns would also increase significantly the reliability of results and should be encouraged. Scaling is a key issue in the application of the proposed method. It is generally useful to choose scaling based on the final match (difference between \(R_{ijd}\) and \(\{c_{i\omega}\hat{\psi}_{ijd}\}\{\hat{\psi}_{ijd}\}^{T}\)) in an optimisation scheme [21] initialised using equation (21).

Another characterization of the reliability can be found by comparing the \(\Gamma_{ijd}\) found for each test. In the present case, the non-proportional damping interactions of different modes is quite often negligible and the separation criterion (16) provides a bound on the

![Graph](image-url)

**Figure 5.** Frequency response function of accelerometer 3 for an input at actuator location 5. ——, measured FRF; ---, non-proportionally damped identified model; (···) proportionally damped identified model.
maximum level of interaction. From this indication modes 5, 6, 7, 8 are very coupled, and modes 3 and 4 to a lesser extent. The off-diagonal terms of $G_{ID}$ for these modes, the only ones with a significant influence, are shown in Table 1 for the six models.

In the test data used, the damping of poles varies by as much as 27%, so one cannot expect the non-proportional damping matrices to be perfectly consistent. The consistency of the results clearly indicates that the experimentally determined values are physical characteristics of the system and not the result of identification inaccuracies which would differ from test to test.

The limitations of the approach appear only in the last two columns, where the effects of test variations and identification errors are too large for a consistent result. A complete analysis of the effects linked to these terms shows that they introduce extremely small modifications of the frequency response and thus cannot be expected to be identified accurately.

The methodology proposed in Section 3 preserves the reciprocity by working on the output matrix only (and defining inputs by duality) but does not constrain the system matrices to be positive-definite (strictly for the mass matrix and semi- for the damping and stiffness matrices for systems with rigid body modes). The positive-definiteness should thus be verified a posteriori, using for example expressions in terms of the proper complex modes $\tilde{\psi}_{ID}A_{ID}\tilde{\psi}_{ID}^T > 0$, $\tilde{\psi}_{ID}A_{ID}\tilde{\psi}_{ID}^T < 0$, and $\tilde{\psi}_{ID}A_{ID}\tilde{\psi}_{ID}^T < 0$ (28)

which correspond to the positive-definiteness of the mass, damping and stiffness matrices respectively and can be easily derived from the inverse expressions (6–7).

In the cases treated here, the mass and stiffness were always found to be positive-definite. However, this might not be true for several reasons: the complex modes residue may not have been well identified; the complex mode outputs may have been improperly scaled (only phase errors can lead to non-positive definite matrices); and the complex modes may be too far from being proper leading to meaningless proper approximations, etc. In such cases, the proposed method does not provide estimates of normal modes. However, the method could be applied on smaller groups of complex modes.

For the lightly damped structures considered here, a non-positive definite damping matrix may result from extremely small errors in the complex modeshapes and, in practice, this does happen quite often, even for models that match the measured frequency responses extremely well. The negative eigenvalues of the damping matrix were, however, quite small. Finally, the proposed method preserves poles. One thus has the guarantee that the identified normal mode model is stable.

<table>
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<th>Terms</th>
<th>$\Gamma_{56}$</th>
<th>$\Gamma_{36}$</th>
<th>$\Gamma_{58}$</th>
<th>$\Gamma_{38}$</th>
<th>$\Gamma_{57}$</th>
<th>$\Gamma_{37}$</th>
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<td>−0.10</td>
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</table>

Table 1

Important experimental off-diagonal values of the modal damping matrix identified independently for each of the six tests (shaker locations)
A method for the identification of normal modes and associated non-proportional damping matrix was proposed. The validity of the method was demonstrated on an experimental example by obtaining, from independent tests, coherent estimates of normal modes as well as off-diagonal terms of the modal damping matrix. Initial identification was done using the IDRC algorithm [16] which may have been a key factor in obtaining good enough complex mode estimates.

It was shown that only proper complex mode models have a normal mode representation, so that the main steps of the method are to select groups of complex modes using the separation criterion and to find a proper approximation of these modes. An important aspect of this approach is that the poles of the normal and complex mode models are identical.

The validity of the proposed transformation is based on the assumption of proportional damping by block which was introduced and motivated. This hypothesis is equivalent to properness and completeness for a group of complex modes. It can also be used for analysis of combined analytical (undamped FE) and experimental models [23] or component mode synthesis with experimental damping models [20]. These applications form a natural extension of the method proposed here and rapidly show the type of problems where it is applicable.

The models used in the paper do not consider the effect of modes that are not within the retained group of complex modes [contributions lumped in an \( E(s) \) term]. For \( NR \) identified conjugate pairs of complex modes one has thus found a second-order model with \( NR \) dof. For a particular choice of \( NR \) dof, one has found associated mass, damping, and stiffness matrices. These matrices can be expressed in any co-ordinate system (choice of dof) but only give an estimate of the kinetic energy, Rayleigh's dissipation and strain energy linked to the identified modes. The relation between these energies and the true energies of the continuous structure, which has an infinite number of modes, is a very interesting issue which was not addressed. The component mode synthesis literature [24] clearly indicates that the residual flexibility (stiffness contribution in the \( E \) term), which is not considered here, is quite essential for many problems.

ACKNOWLEDGEMENT

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REFERENCES

APPENDIX A: NOMENCLATURE

\( [H(s)]_{N \times N} \)  
Frequency response function matrix

\( [b]_{N \times 1} \)  
input, output shape matrices

\( [E]_{N \times 2N} \)  
correction for out-of-band modes

\( M, C, K \)  
\( N \times N \) mass, damping, and stiffness matrices

\( \{q\}_{N \times 1} \)  
Degrees of freedom (dof)

\( \{p\}_{N \times 1} \)  
principal co-ordinates

\( [R]_{N \times N} \)  
residue matrix of the \( j \)th pole

\( \{u(s)\}_{N \times 1} \)  
physical inputs or forces

\( [U_k]_{N \times 1}, \{V_k\}_{N \times 1} \)  
kth left, right singular vectors of a residue matrix

\( \{y(s)\}_{N \times 1} \)  
physical outputs or displacements

\( \phi \)  
matrix of normal modes (columns are \( \phi \))

\( [T]_{N \times N} \)  
normal mode damping matrix (\( T = \phi^T C \phi \))

\( [A]_{2N \times 2N} \)  
diagonal matrix of the 2N poles of the model

\( [\theta]_{2N \times 2N} \)  
first-order scaled complex modeshape matrix

\( \sigma_{jk} \)  
kth singular value of a residue matrix

\( \omega_{j \pm \xi} \)  
jth pole frequency and damping ratio

\( [\Omega]_{N \times N} \)  
diagonal matrix of normal mode frequencies squared

\( \phi^* \)  
matrix of complex modeshapes

\( * \)  
complex conjugate