EFFICIENT SENSITIVITY ANALYSIS BASED ON FINITE ELEMENT MODEL REDUCTION

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Abstract

The computation of frequency and modeshape sensitivities with respect to design parameters is essential to many structural optimization and finite element update algorithms who use this gradient information to orient the search for a minimum of various objective functions. Sensitivity computations may often become prohibitively expensive if large-dimensional models are used. On the other hand, approximating the gradients may lead to poor estimates and a loss of convergence.

The cost of Nelson's exact method to compute modeshape sensitivities is generally not acceptable for industrial size models. The present study thus gives a general categorization of existing approximation methods with suggestions for some new extensions. Iterative corrections of the sensitivities significantly improve the accuracy of predictions found using Fox and Kapoor's modal based sensitivities but still require the computation of the exact modes at the current design point. Fixed basis model reduction allow an extremely fast and relatively accurate prediction of both modeshapes and their sensitivities over a limited area of the parameter space. Illustrations using a 7980 DOF engine block model are provided to demonstrate the applicability of the proposed approaches while giving indications on their cost and accuracy for a model of realistic size.

1 Introduction

Iterative methods are widely used for finite element model updating and structural optimization. Most of these approaches use partial derivatives, called *sensitivities*, of properties with respect to physical parameters of the full order model. Accurate and yet inexpensive evaluations of sensitivities is thus a major issue.

Computation of eigenvalue and eigenvector sensitivities has been the object of an extensive literature. The modal method (Fox and Kapoor [1]) is most widespread although its relatively poor accuracy is well known. A method to compute the exact solution was proposed by Nelson [2] but the present study will illustrate that the associated computational cost is too large for industrial models. As alternatives to the exact method, Ojalvo and Zhang [3] proposed to use a basis of Lanczos vectors to replace the modes used by Fox and Kapoor while Wang [4] and in a more general setting Liu [5] proposed the use of static corrections to the modal method.

The present study gives a presentation of all these methods within the unified framework of fixed and variable basis model reduction. This framework allows the simple introduction of new approaches to obtain low cost predictions of both modeshapes and their sensitivities over an arbitrary segment of design parameter space.

The theoretical presentation is followed by a detailed analysis of the various methods in terms of accuracy and computational cost for a 7980 DOF finite element model of an engine block [6]. It is assumed that this model size will provide real insight on the applicability of the methods to industrial problems while still allowing the computation of an exact solution for accuracy evaluations.

2 Theoretical aspects

2.1 Exact solution

Modes are solution of the eigenvalue problem

$$\left[K(p) - \omega_j^2 M(p)\right]\{\phi_j\} = [Z(\omega_j, p)]\{\phi_j\} = \{0\} \tag{1}$$

and verify two orthogonality conditions with respect to mass

$$\{\phi_k\}^T[M]\{\phi_j\} = \delta_{jk} \tag{2}$$

and stiffness

$$\{\phi_k\}^T[K]\{\phi_j\} = \omega_j^2\delta_{jk} \tag{3}$$

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The mass normalization of mode j, linked to the constant $\{\phi_j\}^T[M]\{\phi_j\}$, is arbitrary and will be assumed to be equal to 1 in all cases (as shown in the orthogonality conditions (2)-(3)).

Equation (1) being valid for all values of p, its derivative with respect to p is also equal to zero, which one easily shows to result in

$$[Z(\omega_j)]\left\{\frac{\partial\{\phi_j\}}{\partial p}\right\} = \{B(\omega_j)\} \tag{4}$$

where

$$B(\omega_j) = \left[-\frac{\partial K}{\partial p} + \frac{\partial \omega_j^2}{\partial p}M + \omega_j^2\frac{\partial M}{\partial p}\right]\{\phi_j\} \tag{5}$$

By definition of modes (1), the dynamic stiffness $[Z(\omega)]$ is singular at modal frequencies ω_j so that that equation (4) does not necessarily have a solution. The kernel of $[Z(\omega)]$ is however $\{\phi_j\}$ and it is a well known theorem of linear algebra that equations of the form $Zq = B$ with Z singular have solutions if and only if B is orthogonal to the kernel of Z^T. Thus here, one must have $\{\phi_j\}^T B(\omega_j) = 0$ which defines the sensitivity of modal frequencies

$$\frac{\partial \omega_j^2}{\partial p} = \{\phi_j\}^T\left[\frac{\partial K}{\partial p} - \omega_j^2\frac{\partial M}{\partial p}\right]\{\phi_j\} \tag{6}$$

Again it is known from linear algebra that solutions of (4) take the general form $\partial\{\phi_j\}/\partial p = \psi_j + \alpha\phi_j$ where ψ_j is an arbitrary particular solution of $[Z(\omega_j)]\{\psi_j\} = \{B(\omega_j)\}$.

As proposed by Nelson [2], a particular solution of (4) can be determined by imposing that one of the components of ψ_j to be equal to zero. This particular solution clearly exists as long as the corresponding component of ϕ_j is non-zero. Knowing that a component of ψ_j is zero, one can eliminate a row and a column of (4) which leads to a non-singular set of equations that can be solved relatively easily. This solution however requires the factorization of a block of $Z(\omega_j)$. This factorization must be performed at the frequency of each of the desired modeshape sensitivities which tends to be prohibitively expensive for realistic finite element models (see the numerical application later).

Finally a condition is needed to define the coefficient α in the general form of the solution. Assuming that the modeshape is always mass normalized as shown in (2), this condition can be derived with respect to p which leads to

$$\{\phi_j\}^T M\frac{\partial\{\phi_j\}}{\partial p} = -\frac{1}{2}\{\phi_j\}^T\frac{\partial M}{\partial p}\{\phi_j\} \tag{7}$$

Thus, given ψ_j a particular solution of $Z(\omega_j)\psi_j = B(\omega_j)$, the sensitivity of the mass normalized modeshapes is given by

$$\frac{\partial\{\phi_j\}}{\partial p} = \{\psi\} - (\phi_j^T M\psi + \frac{1}{2}\phi_j^T\frac{\partial M}{\partial p}\phi_j)\{\phi_j\} \tag{8}$$

Note that for cases with multiple modes, this discussion needs further considerations as found in Refs. [7, 8, 9]. All the methods considered in this paper could however be extended to treat multiple modes.

2.2 Approximations by projection of the solution

Projection methods are widely used to seek approximations of the properties of dynamic systems. The simplest of such approximations is the projection on a truncated modal basis. Condensation [10], component mode synthesis and substructuring methods [11], approximations on series of Krylov or Lanczos vectors [12] are other well known methods (the later are typically used to seek approximations of low frequency eigenvalues [13]).

All these methods are linked to the assumption that an accurate approximation of the response can be found in a subspace spanned by the columns of a rectangular projection matrix T (with N rows and $NR << N$ columns). As analyzed in Ref. [14], a constant basis can be used to approximate the solutions of a family of models characterized by the parameters p. The approximate modes of a model projected on the basis T are given by $\phi_j = T\phi_{jR}$ with ϕ_{jR} solution of the projected (reduced) eigenvalue problem

$$[T]^T[K(p) - \omega_{jR}^2 M(p)][T]\{\phi_{jR}\} = \{0\} \tag{9}$$

Assuming that T is fixed, equation(9) is valid for all values of p and can be derived as done for the full order model in the previous section. The general form of the approximate sensitivity is clearly given by

$$\frac{\partial\{\phi_j\}_R}{\partial p} = [T](\psi_{jR} + \alpha\phi_{jR}) \tag{10}$$

where ψ_{jR} is solution of

$$[Z_R(\omega_j)]\psi_{jR} = \{B_R(\omega_j)\} \tag{11}$$

with $Z_R = T^T ZT$ and

$$B_R = T^T\left[-\frac{\partial K}{\partial p} + \frac{\partial \omega_j^2}{\partial p}M + \omega_j^2\frac{\partial M}{\partial p}\right]T\{\phi_{jR}\} \tag{12}$$

The method proposed by Fox and Kapoor [1] is the most widespread projection method used to approximate modeshape sensitivities. At any design point p, the projection basis is taken to be a truncated set of the exact modes at this design point $T = [\phi_1(p)\ldots\phi_{NR}(p)]$.

By multiplying equation (4) on the left by T^T and using the orthogonality conditions (2)-(3), one classically shows that a particular solution of the projected equation is given by

$$\psi_{jR} = \sum_{k \neq j} \frac{\{\phi_k\}^T \left[\frac{\partial K}{\partial p} - \omega_j^2 \frac{\partial M}{\partial p} \right] \{\phi_j\}}{\omega_j^2 - \omega_k^2} \{\phi_k\} \qquad (13)$$

and that the component of the exact solution in the direction of ϕ_j is given by $\alpha = -\phi_j^T (\partial M \partial p) \phi_j / 2$.

Rather than using these expressions that are only valid for the exact modes at the current design point, one should realize that they correspond to the use of Nelson's exact method applied on the model projected on the associated basis. Nelson's method being applicable to any reduced sensitivity equation of the form (11) finding explicit expressions for the particular solution is not useful. Furthermore the cost associated to this evaluation is negligible since the dimension of the subspace (number of columns of T) is small compared to the size of the initial model. When designing improved methods for the approximation of sensitivities, the effort should thus concentrate on building a reduction basis that will give accurate predictions of the sensitivities.

The truncated modal basis, while generally available is not the most efficient reduction basis in terms of allowing accurate predictions of modeshape sensitivities. As a first example of alternate bases, Ojalvo and Wang [3] realized that the estimates of $\phi_j(p)$ are often determined by projection of the model on a basis of Lanczos vectors which span the same subspace as the Krylov vectors given by $T_p = (K^{-1}M)^{p-1} K^{-1} T_0$ and thus proposed a method allowing the use of the same basis of Lanczos vectors for the estimation of modeshapes and modeshape sensitivities.

This approach continues to accept the cost of computing the exact modes at each iteration but uses a larger projection basis to estimate sensitivities, so that results can be expected to be more accurate. The applications in section 3 will however show that better results are obtained when some knowledge of the modification is taken into account as shown in the following sections.

2.3 Iterative determination of the exact solution

A first category of methods continues to accept the cost of computing the exact modes at each iteration but seeks to find a way of approximating the exact sensitivity. Such improvements will be found by complementing the modal basis of Fox's method or the Lanczos basis of Ojalvo's method by additional vectors that take properties of the modification into account.

Realizing that in the sensitivity equation (4), the second member $B(\omega_j)$ corresponds to a load that is representative of the modification, it is useful to complement the basis T by the static responses to this load

$$T_C = \left[\begin{array}{cc} \{\phi_j(p)\} & \left[\tilde{K}\right]^{-1} [B(\omega_1, p) \dots B(\omega_n, p)] \end{array} \right] \qquad (14)$$

As shown here, the augmented basis should include the static responses to modification loads of several modes (the one wishes to compute the sensitivity of) rather than, as proposed in Ref. [4], consider a static correction for each mode. As mentioned in Ref. [5], a mass shifted stiffness matrix $\tilde{K} = K + \lambda M$ can be used in place of the nominal stiffness matrix when rigid body modes pose problems. The static flexible response would be another alternative (see section 6.8 in Ref. [13] on iterations in presence of rigid body modes).

This first level correction can be extended using the following vectors of the Krylov series $T_k = \left[\tilde{K}^{-1} M\right]^k \left[\tilde{K}\right]^{-1} B_{1 \dots n}$ but care must be taken to orthogonalize the successive additions T_k to the base subspace spanned by T_C. One could for example use the Lanczos orthogonalization scheme for this purpose. This would go back to the idea proposed by Ojalvo and Wang [3] but use $\left[\tilde{K}\right]^{-1} [B(\omega_1, p) \dots B(\omega_n, p)]$ as the base vectors to restart a block Lanczos algorithm.

Note that one saves a lot of time by using the same factorization of K or $\tilde{K}$ to compute eigenvectors (using a subspace or Lanczos method) and the corrections (14) needed to compute accurate sensitivities.

2.4 Fixed basis approximation of modes and sensitivities

The idea that a fixed Ritz basis can be used to compute the modes for various values of the parameters was detailed in Ref. [14]. Among the various methods proposed to build such fixed basis approximations, one will here consider a multi-model basis which combines modes computed at both ends of an interval

$$T_M = \left[\begin{array}{cc} \{\phi_j(p_1)\} & \{\phi_j(p_2)\} \end{array} \right] \qquad (15)$$

and a basis combining modes and sensitivities at the initial point of the interval

$$T_S = \left[\begin{array}{cc} \{\phi_j(p_1)\} & \left. \frac{\partial \{\phi_j\}}{\partial p} \right|_{p_1} \end{array} \right] \qquad (16)$$

The motivation for using such approaches is that the reduced model thus created will be able to predict both the modes and their sensitivities at a minimal cost but with a relatively good accuracy. While the two methods considered in this study are clearly tailored for line searches (predictions over a segment of parameter space), they could also be applied to parameter areas of higher dimensions. The size of the basis would however increase fairly rapidly with the number of independent directions in parameter space, so that the interest in terms of cost reduction would decrease rapidly.

3 Comparisons in terms of cost and precision

3.1 Sample problem

To evaluate the different methods, the case of the engine block model shown in figure 3.1 and analyzed in Ref. [6] is considered. This model contains 2660 nodes, 7980 DOFs, 1380 HEXA8 solid elements. The connectivity is fairly high so that the Cholesky factor after reverse Cuthill-McKee renumbering contains 2,848,742 non-zero elements (density of 4.4%).

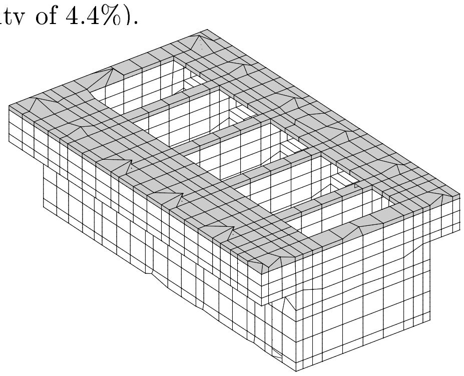

Figure 1: 7980 DOF engine block model used for the detailed accuracy evaluation.

The elastic modulus of the top part of the engine block (shown in gray in the figure) is used as a design parameter as might be done in an optimization study on the properties of this model. Results shown later in the section are given with the current modulus being a fraction (between 0.1 (90% decrease) and 2 (100% increase)) of the nominal value.

Numerical comparisons are be given for the prediction of the first 5 flexible modes and their sensitivities. As the structure is free-free, there are always 6 rigid body modes but those are invariant and are predicted exactly by all methods.

The multi-model method MUL (15) is used with 6 rigid body modes and the first five flexible modes at both ends of the considered parametric interval $E \in [0.1\ 2]E_0$.

The nominal + sensitivity method N+S (16) is used with 6 rigid body modes, the first five flexible modes and their sensitivities at $E = 0.4 * E_0$.

MUL and N+S thus both use bases with 16 vectors. The variable basis methods are used with more vectors (20 for FOX and 40 for OJA) since using them with 16 modes gives very poor results that are harder to present on the plots.

3.2 Structure of computational costs

Table 1 shows a decomposition of computational costs associated with the different methods with the major steps being

- D — Cholesky decomposition of the stiffness or dynamic stiffness matrix, with specific precautions taken when the matrix is singular or not positive definite.
- I — forward/backward substitution to solve a problem of the form $[Z]\{q\} = \{F\}$ with the $R^T R$ or LDL^T decomposition of Z given.
- EV — Cost of full order eigenvalue solution. Using the Lanczos algorithm this cost is driven by the decomposition D and a series of two times the number of modes forward/backward substitutions and orthogonalizations with respect to the mass matrix.
- ER — Reduced eigenvalue problem
- n_S — number of modeshape sensitivities computed.
- n_L — number of points evaluated for a line search.

Table 1: Approximate cost structure associated to a n_L point line search with n_S sensitivities at each point

Exact (Nelson)		$n_L(EV + n_S(D + I))$
Fox and Kapoor	FOX	$n_L(EV + n_S(P))$
Ojalvo,Wang	OJA	$n_L(EV + n_S(4P)$
Fox+correction	F+C	$n_L(EV + n_S(I + P))$
Multi-model	MUL	$2EV + P + n_L(ER)$
Nominal + sensit.	N+S	$EV + n_S(D + I) + n_L(ER)$

The different methods were implemented in the environment provided by the Structural Dynamics Toolbox for use with MATLAB [15] and table 2 compares actual costs on a Silicon Graphics Octane workstation with 256 MB physical memory. Care was taken to use efficient algorithms but, as always, evaluations of computational costs could be somewhat modified by further optimization of the algorithms, change of computer or of computational environment. Known significant distortions of cost are found for

- *the exact method.* MATLAB provides a sparse Cholesky rather than LDL^T decomposition so that for non-positive definite matrices a significantly more expensive LU decomposition is used. A change in this strategy might decrease the cost of the factorization in the exact sensitivity computation by at least a factor 3 but nothing close to the factor 7000 needed to obtain similar to the F+C method.

- *the N+S reduced model.* This method projects on a basis of exact sensitivities. Since it will be shown that the F+C method is very accurate, initial point sensitivities computed with this method would drastically decrease the initial cost while not decreasing the accuracy very much.

Table 2: CPU times (in seconds) associated to the computation of sensitivities for the engine block application

Exact	$n_L \times (236 + n_S \times (687 + 2.5))$
Fox,Kapoor	$n_L \times (236 + n_S \times (0.1))$
Ojalvo,Wang	$n_L \times (236 + n_S \times (0.1)$
Fox+correction	$n_L \times (236 + n_S \times (2.6))$
Multi-model	$2 \times 236 + n_L \times (0.0)$
Nominal + sensitivity	$236 + n_S \times (690) + n_L \times (0.0)$

Table 2 clearly illustrates that the exact approach is particularly costly which is the main motivation for this study. NEL and OJA have very low costs but do not take modifications into account which limits their accuracy (see details in the next section).

F+C seems to provide an extremely efficient compromise provided that the cost of the full order eigenvalue solution is acceptable and that the same factorization of the stiffness matrix can be used for both the eigenvalue and sensitivity computations (the cost of the factorization is more than half the total cost of the full order eigenvalue solution). The first order correction seemed sufficient here but higher order corrections, seen in section 2.3, would not increase the cost very much.

The fixed basis reduction methods MUL and N+S provide very interesting alternatives to the need to recompute exact modes at each design point. But they can only be constructed to provide good approximations for parameter segments or possibly low dimension boxes within the full parameter space. For traditional multiparameter optimization algorithms, they would thus be mostly of interest for detailed line searches using $n_S > 2$ points.

3.3 Modeshape and frequency sensitivities with fixed basis methods

The fixed basis reduction methods (here MUL and N+S) give approximate predictions of modeshapes and frequency sensitivities at a negligible cost. A good understanding the accuracy of these predictions is clearly needed to properly analyze the quality of modeshape sensitivity predictions.

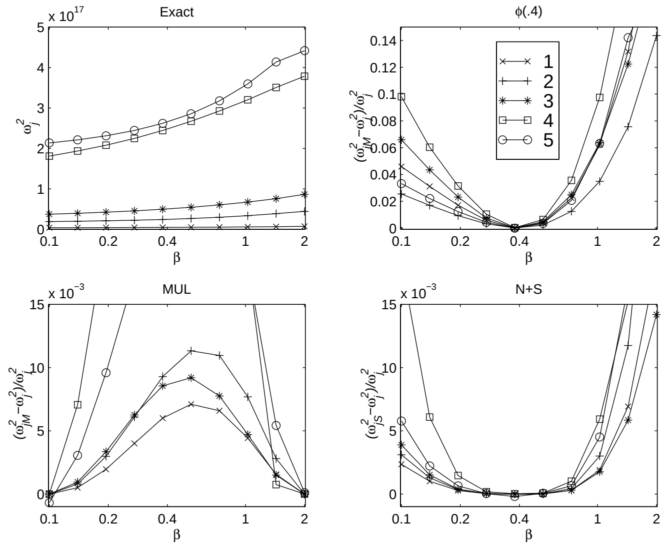

Figure 2: Evolution of *frequencies* and relative error on frequencies predicted by reduced models (16 modes computed for $\beta = .4$, MUL, and N+S)

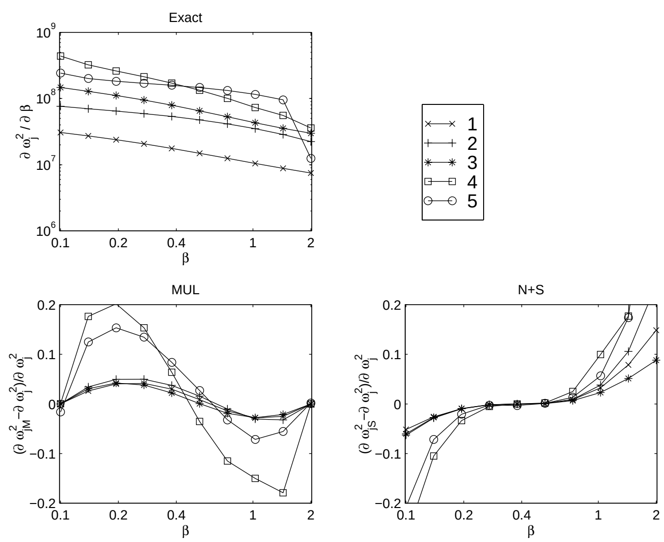

Figure 3: Evolution of *frequency sensitivities* and relative error on frequency sensitivities predicted by reduced models

The evolution of frequencies on the interval of variations considered for the top plate modulus is shown in figure 2. The relative error on the frequencies also shown in the figure clearly indicates the advantages of the two reduction methods retained here. The MUL reduction is, by definition, exact at both ends of the interval, hence a bell shaped error curve with a maximum below 4% for all modes. The sensitivities added to the N+S reduction lead to a very good accuracy for a wider region near the initial $E = 0.4E_0$ point but to significant divergence later on. Finally, the predictions obtained using a projection on 16 modes computed at $E = 0.4E_0$ gives relative errors on frequency predictions that are higher by a factor 10 clearly indicating the interest of alternate fixed basis methods.

The methods using exact modes at the current design point predict the exact frequency sensitivities shown in figure 3. The approximate predictions by MUL and N+S follow the trends seen for frequency predictions with maximum relative errors on sensitivity predictions below 20%. Note that mode 5 strongly interacts with mode 6 above $\beta = 1.5$ so that pairing errors can be expected (see difficulties in figures 4 and 6).

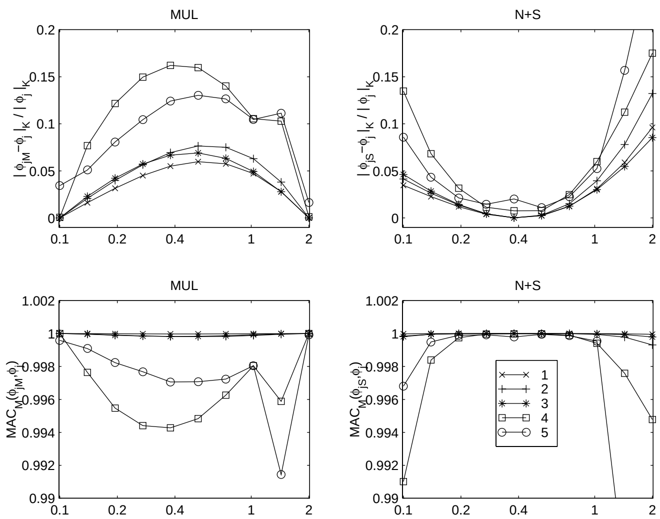

Figure 4: $rel_K(\phi_{jR}, \phi_j)$ and MAC_M for predictions of modeshapes using the MUL (15) and N+S (16) reduction bases

For modeshape comparisons the mass weighted Modal Assurance Criterion

$$\mathrm{MAC}_M(\phi_{jR}, \phi_j) = \frac{\|\phi_{jR}^T M \phi_j\|^2}{(\phi_{jR}^T M \phi_{jR})(\phi_j^T M \phi_j)} \tag{17}$$

is common but gives a relatively forgiving measure of correlation between shapes, while the relative strain energy error

$$rel_K(\phi_{jR}, \phi_j) = \frac{(\phi_{jR} - \phi_j)^T K (\phi_{jR} - \phi_j)}{\phi_j^T K \phi_j} \tag{18}$$

is much more accurate (values below 0.2 indicate very good correlation).

Figure 4 clearly shows the very high accuracy of modeshape predictions by those two methods with similar ranges of validity as those seen for frequency predictions.

3.4 Modeshape sensitivity predictions

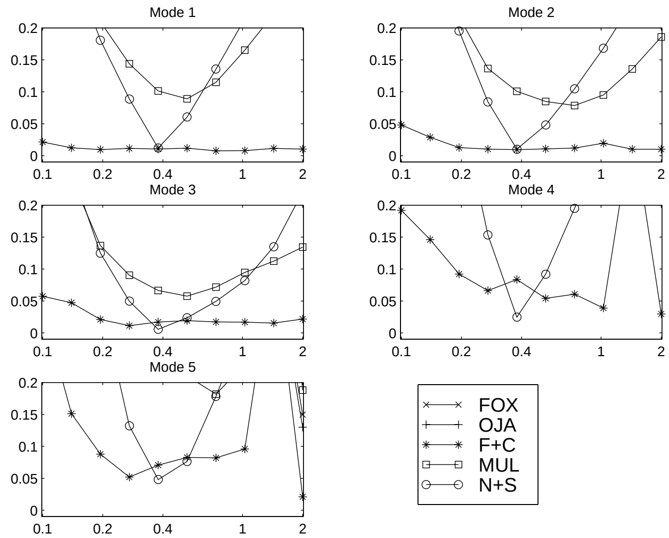

Figure 5: Evolution of $rel_K(\partial\phi_{jR}/\partial p, \partial\phi_j/\partial p)$ for the first 5 flexible modes and the different methods

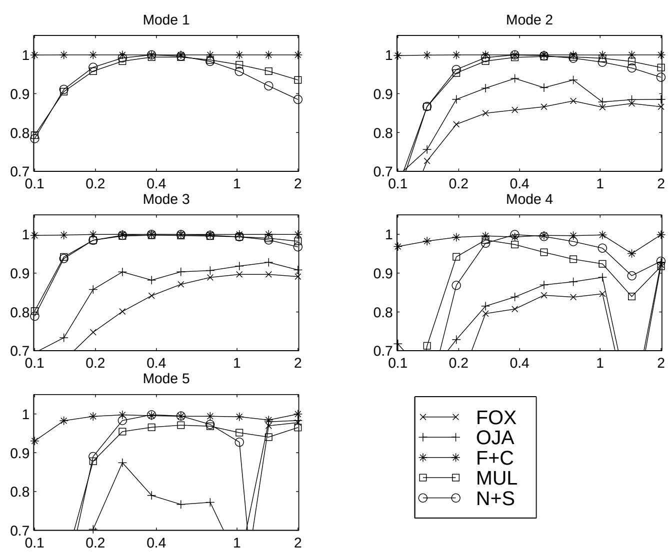

Figure 6: Evolution of $\mathrm{MAC}_M(\partial\phi_{jR}/\partial p, \partial\phi_j/\partial p)$ for the first 5 flexible modes and the different methods

Figures 5-6 give similar indications on the accuracy of modeshape sensitivity predictions by the 5 methods considered here. Both FOX and OJA give very poor results (even though projections on bases of 20 and 40 vectors are considered while the fixed basis methods use only 16). The F+C method is always extremely accurate. A second correction would only help for $E < 0.2E_0$ where the accuracy is already quite good.

The fixed basis methods give very encouraging results. The bell shaped curve found for modeshape predictions using the MUL method is now inverted with the best sensitivity predictions obtained near the middle of the interval. The two measures of error are clearly not equivalent since the MAC tends to show the MUL and N+S methods to be equivalent while the relative strain energy error gives and advantage to the N+S method.

4 Conclusions

A general classification of existing methods for modeshape sensitivity computations has been provided with suggestions for new extensions of existing methods. In particular it was highlighted that Nelson's exact method provides an extremely general procedure to estimate sensitivities for arbitrary reduction bases.

Comparisons in terms of numerical cost and accuracy clearly indicate that Nelson's exact method is generally too expensive while first or second order static corrections to the modal method give extremely accurate results at fractions of the cost.

When the exact computation of modes at each design point is too costly, the fixed basis methods proposed in Ref. [14] were shown here to allow low cost predictions of both modeshapes and sensitivities over limited domains of parameter space. These approaches should be extremely helpful in iterative optimization algorithms that use multiple point line search sequences.

Acknowledgments

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