Damping and complex modes.

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Outline

Modes real and complex
Spectral decomposition
When are modes real?
Proportional damping
Real from complex test modes
Complex non modes

Damping modeling
Material, Joint, ...
Damped FEM models
Local/system model
Modes
Spectral decomposition
When real?
Proportional damping
Real from complex
Complex non modes
Damping
Material, Joint, ...
Damped models
Local/system model

Mode ≈ 1 DOF system

\[ m\ddot{q}(t) + c\dot{q}(t) + kq(t) = F(t) \]

\[ H(\omega) = \frac{q(\omega)}{F(\omega)} = \frac{1}{-\omega^2 m + i\omega c + k} \]
\[ H(s) = \frac{1}{s^2m + cs + k} = \frac{1}{m} \left( \frac{\beta}{s - \lambda} + \frac{\bar{\beta}}{s - \bar{\lambda}} \right) \]
1 DOF : frequency domain

\[ H(s) = \frac{1}{s^2m + cs + k} = \frac{1}{m} \left( \frac{\beta}{s - \lambda} + \frac{\bar{\beta}}{s - \bar{\lambda}} \right) \]

\[ \beta = \frac{1}{i\omega \sqrt{1 - \zeta^2}} \]

\[ \lambda = -\zeta \omega_n \pm i\omega_d \quad , \quad \omega_d = \omega_n \sqrt{1 - \zeta^2} \]

\[ \omega_n = \sqrt{k/m} \quad , \quad \zeta = \frac{c}{2\sqrt{km}} \]

1 DOF system (single mode for mechanical system) has
2 complex conjugate poles (linear system modes)
MDOF SISO system

Spectral decomposition

MDOF more than 1 pole
SISO Rj is 1x1

\[ \sum_{j \in \text{identified}} \left( \frac{[R_j]}{s - \lambda_j} + \frac{[\bar{R}_j]}{s - \bar{\lambda}_j} \right) \]
MDOF MIMO system

- Sames poles for each IO pair
- Residue matrix in spectral decomposition
Experimental modes are often “real”

“Real modes” Residues for I/O pairs line up
“Complex modes” Residues have a phase spread
Poor modes Have complex residues
Modes of linear system

- **Nominal state-space form**

\[
\begin{align*}
\{ \dot{x}(t) \} &= [A] \{ x(t) \} + [B] \{ u(t) \} \\
\{ y(t) \} &= [C] \{ x(t) \} + [D] \{ u(t) \}
\end{align*}
\]

- **Left and right eigenvalue problems**

\[
\begin{align*}
[A] \{ \theta_j R \} &= \lambda_j \{ \theta_j R \} \\
\{ \theta_j L \}^T [A] &= \{ \theta_j L \}^T \lambda_j
\end{align*}
\]
Modes of linear system

- Mode shape orthogonality and scaling conditions
  \[
  [\theta_L]^T [A] [\theta_R] = [\Lambda] \quad \text{and} \quad [\theta_L]^T [\theta_R] = [I]
  \]

- Diagonal state-space model
  \[
  \begin{align*}
  \{p\}_s &= [\Lambda] \{p\} + [\theta_L^T B] \{u(s)\} \\
  \{y(s)\} &= [C \theta_R] \{q(s)\} + [D] \{u(s)\}
  \end{align*}
  \]

  Mode shape  Participation factor
Modal coordinates, state-space

- Inverting diagonal state-space leads to the spectral decomposition

\[ H(\omega) = \sum_{j=1}^{N} \left( \frac{[R_j]_{NS \times NA}}{i\omega - \lambda_j} + \frac{[\bar{R}_j]_{NS \times NA}}{i\omega - \lambda_j} \right) \]

\[[R_j]_{NS \times NA} = \{[C]\{\theta_jR\}\}_N{S \times 1} \{[\theta_jL]^{T}[B]\}_{1 \times NA}\]

- Residues of linear systems have no reason to have a single phase ("be real")
Normal modes of elastic structure

- Nominal model (elastic + viscous damping)

\[
\begin{bmatrix}
M s^2 + C s + K
\end{bmatrix}
\{q(s)\} = \begin{bmatrix} b \end{bmatrix}\{u(s)\}
\{y(s)\} = \begin{bmatrix} c \end{bmatrix}\{q(s)\}
\]

- Conservative eigenvalue problem

\[-[M] \{\phi_j\} \omega_j^2 + [K]_{N \times N} \{\phi_j\}_{N \times 1} = \{0\}_{N \times 1}\]

- \( M > 0 \) & \( K \geq 0 \) \( \Rightarrow \) \( \phi \) real
Normal modes of elastic structure

- Orthogonality
- Scaling conditions
  - Unit mass
  - Unit amplitude
- Principal coordinates

\[
\begin{bmatrix}
\phi^T M \phi \\
\phi^T K \phi
\end{bmatrix} =
\begin{bmatrix} \mu_j \\
\mu_j \omega_j^2
\end{bmatrix}
\]

\[
\{ \phi_j \}^T M \{ \phi_j \} = 1
\]

\[
[c_s] \{ \tilde{\phi}_j \} = 1 \quad \mu_j (c_s) = (\{ c_i \} \{ \phi_j \})^{-2}
\]

\[
\begin{bmatrix}
[I] s^2 + [\Gamma] s + \begin{bmatrix} \omega_j^2 \end{bmatrix}
\end{bmatrix} \{ p(s) \} = \begin{bmatrix} \phi^T b \end{bmatrix} \{ u(s) \}
\]

\[
\{ y(s) \} = [c\phi]\{ p(s) \}
\]
Modal damping assumption

- Assume $\Gamma$ diagonal

$$[\Gamma] = [\phi^T C \phi] = \begin{bmatrix} \begin{array}{c} 2\zeta_j \omega_j \end{array} \end{bmatrix}$$

- Leads to second order spectral decomposition

$$H(s) = \sum_{j=1}^{N} \frac{[c] \{\phi_j\} \{\phi_j\}^T [b]}{s^2 + 2\zeta_j \omega_j s + \omega_j^2} = \sum_{j=1}^{N} \frac{[T_j]}{s^2 + 2\zeta_j \omega_j s + \omega_j^2}$$

Mode shape  Participation factor  Residue
Modal damping assumption

This is the only widespread damping model

Why?

• Compellingly practical
• Easy combination of test and analysis
• Sufficient mathematical conditions
  • Rayleigh $[C] = \alpha[M] + \beta[K]$
  • Caughey $[C] = \sum \alpha_{k,l} [M]^k [K]^l$

Modal also called proportional damping

• Often induces small modifications in behaviour
Spectral decompositions

- General linear system

\[
\begin{bmatrix} [R] \\ \end{bmatrix} + \frac{[\bar{R}]}{s - \lambda} = 2 \left( s \text{Re}(R) \right) + \frac{-\text{Re}(\lambda) \text{Re}(R) - \text{Im}(\lambda) \text{Im}(R)}{s^2 - 2(\lambda + \bar{\lambda})s + \lambda \bar{\lambda}}
\]

- Structure with modal damping

\[
\begin{bmatrix} T \end{bmatrix} = \frac{T/(i\text{Im}(\lambda))}{s^2 + 2\zeta \omega s + \omega^2} = \frac{T/(i\text{Im}(\lambda))}{(s - \lambda)} + \frac{T/(i\text{Im}(\bar{\lambda}))}{(s - \bar{\lambda})}
\]

Modal damping $\iff$ $R$ is imaginary
When are modes real / complex?

Vibrating structures that are elastic, linear, and time invariant have real modes.

Complex modes are found for

- Damped structures
- Non linear systems
- Time varying systems
- Periodic structures

Non modal

Resonances rather than modes

Mathematical trick

When are complex modes nearly real?
• One tip damper with variable C
• 1 pole moves
• modes remain nearly real
A case with complex modes

- $C = 1 \times 10^3$ N/m/s
- $K_1 = 1.105 \times 10^5$ N/m
- $K_2 = 1.095 \times 10^5$ N/m
- $\theta = 0, 20, 30, 40^\circ$

- Very close frequencies
- Damping and stiffness are not proportionnal
Losing mode complexity

- C = 1e3 N/m/s
- K1,2 = 1.1 + dk N/m
- K2 = 1.1 - dk N/m
- θ = 22°

Different values of frequency separation \( \Rightarrow \) Modes nearly real
When are complex modes nearly real?

- Coupling of two modes by viscous damping

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} s^2 + \begin{bmatrix}
\gamma_{11} & \gamma_{12} \\
\gamma_{21} & \gamma_{22}
\end{bmatrix} s + \begin{bmatrix}
\omega_1^2 & 0 \\
0 & \omega_2^2
\end{bmatrix} \begin{bmatrix}
p_1 \\
p_2
\end{bmatrix} = \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix} \{u(s)\}
\]

\[
p_1 = (1 + e_1)^{-1} \frac{b_1 u}{(s^2 + \gamma_{11}s + \omega_1^2)} + e_2
\]

\[
e_1 = \frac{\gamma_{12}\gamma_{21}s^2}{(s^2 + \gamma_{11}s + \omega_1^2)(s^2 + \gamma_{22}s + \omega_2^2)}
\]

\[
e_2 = \frac{\gamma_{12}s b_2 u}{(s^2 + \gamma_{11}s + \omega_1^2)(s^2 + \gamma_{22}s + \omega_2^2)}
\]

Mode 1 response
Modal damping
Perturbations for non-modal damping
When are complex modes nearly real?

- Uncoupling criterion (Hasselman) $e_i << 1 \iff$

\[
\frac{\min(\zeta_1 \omega_1, \zeta_2 \omega_2)}{|\omega_1 - \omega_2|} \ll 1
\]

corresponds to non overlap of peaks

- Proof based on damping matrix positiveness

\[
\gamma_{12} \gamma_{21} / (\gamma_{11} \gamma_{22}) < 1
\]

- Generalization uncoupling by block (Balmes 97)
Two mode coupling

- Influence on zero & RMS response
- Influence on modeshape complexity
- Effect only significant for high damping

\[ \gamma_{12} \text{ min, 0, max} \]
Modal damping is a good assumption:

- **Provided** low modal overlap

\[ \min(\zeta_1 \omega_1, \zeta_2 \omega_2) / |\omega_1 - \omega_2| \ll 1 \]

- Errors on predicted levels are small
- Assuming real modes is then OK
Experimental modes are often “real”

“Real modes” Residues for I/O pairs line up
“Complex modes” Residues have a phase spread
Poor modes Have complex residues
Real from complex

\[
\lim_{s \to \infty} H(s) = \lim_{s \to \infty} [c]\left[Ms^2 + Cs + K\right]^{-1}[b] = O(1/s^2)
\]

- Complex modes of second order system verify

\[
\sum_{j=1}^{2N} \tilde{\psi}_j \tilde{\psi}_j^T = \tilde{\psi}_{N \times 2N} \tilde{\psi}_{N \times 2N}^T = [0]_{N \times N}
\]

- If properness condition verified

\[
M = \left(\tilde{\psi} \Lambda \tilde{\psi}^T\right)^{-1}
\]

\[
K = \left(\tilde{\psi} \Lambda^{-1} \tilde{\psi}^T\right)^{-1}
\]

\[
C = -M \tilde{\psi} \Lambda^2 \tilde{\psi}^T M
\]
Real from complex

MIT SERC Active control testbed
28 sensors, 6 independent shaker locations
Balmès (PhD 93, MSSP 97)
Real from complex

- $\Gamma$ consistent from test to test
- Proportional damping response significantly different

<table>
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• Real data is almost never that clean
• Use of $\Gamma_{\text{test}}$ difficult and does not change design
Thus for modal test derived damping
• assume modal damping
• if you really want real modes
  – Take the imaginary part of the residue
  – Use appropriation (Foltete 98)
  – Use transformations (Niedbal 84, Zhang 85, Wei 87, Imregun 93, Balmes 93 97, Ahmadian 95, …)
Complex non modes I

Frequency shifts in batch tests induce complexity
Complex non modes II

Simulation: frequency change of 0.2%

Frequency shifts in batch tests induce complexity
Modes
Spectral decomposition
When real?
Proportional damping
Real from complex
Complex non modes
Damping
Material, Joint, ...
Damped models
Local/system model

Computational complex shapes

Propagating waves
• boundary elements
• cyclic symmetry

\[ \{q(t)\} = \text{Re}\left( (\{\phi_1\} + i\{\phi_2\}) e^{i\omega t} \right) = \{\phi_1\} \cos(\omega t) + \{\phi_2\} \cos(\omega t + \pi/2) \]
Modes
- Spectral decomposition
- When real?
- Proportional damping
- Real from complex
- Complex non modes

Damping
- Material, Joint, ...
- Damped models
- Local/system model

Complex shapes

- Non modes
  - Non linear response
  - Poor identification
  - Non invariance of test article
  - Signal processing distortions
  - Operational deflection shapes, propagating waves
  - ...

- Mathematical trick (cyclic symmetry)
- True complex modes (damped linear system)
Modeling damping

- Local models (joints and materials)
- Finite element models, damping design tools
- Simplifying assumptions for system dynamics (dynamic behaviour rather than local knowledge models)
Modes
Spectral decomposition
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Damping
Material, Joint, ...
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Joints

$$F = f(q,t)$$
$$F = k(s)q(s)$$

Non linear models : only practical if local
\[ \sigma = \text{Re}(E_s (1 + i\eta) e^{i\omega t}) \]
**Viscoelastic constitutive relations**

- Stress is a function of strain history
- Complex modulus in Laplace domain

\[ \sigma(s) = E(s, T, \sigma_0) \varepsilon(s) = (E' + iE'') \varepsilon(s) \]
Reduced frequency nomograms
Viscoelastic constitutive laws

Alternatives:

- More relaxation constants
- Fractional derivatives
- Direct use of experimental master curve
Material damping models

Stress/strain curve

Simple models
Consitutive model order

1 pole model (3 parameter)
loss factor is wrong

3 pole model: better match
in band but
not very good outside

Good models
require high order
Other sources of dissipation

• Non linearities
  – Material (plasticity, …), joint
  – Contact (friction dampers, joint damping, micro-slip, …)

• Coupling with other media
  – Radiation in air, water, soil, etc.
  – Gyroscopic damping
  – Electrical systems (active control)
  – Particle filled cavities
  – Lubrication, …
Frequency dependent models

- Dynamic stiffness: linear combination of fixed matrices
  \[
  [Z(E_i, s)] = [Ms^2 + Ke + \sum_i E_i(s, T, \sigma_0) \frac{K_{vi}(E_0)}{E_0}]
  \]

- Direct frequency response
  \[
  [Z(E_i, s)]\{q\} = \{F(s)\}
  \]

- Non-linear eigenvalue extraction
  \[
  [Z(E_i, \lambda_j)]\{\psi_j\} = \{0\}\]
Frequency independent models

- Trick increase model order to gain frequency independence

- Material formulation with internal fields (rational and fractional derivates)

\[ E(s) = E_\infty - \left( \sum_{j=1}^{n} \frac{E_j}{s + \omega_j} \right) \]

\[ q_{vj} = -\frac{E_j}{(s+\omega_j)}q \]

\[
\begin{bmatrix}
M & 0 & 0 \\
0 & M & 0 \\
0 & 0 & M
\end{bmatrix}
\begin{bmatrix}
0 & -M & 0 \\
K_e + E_\infty K_v & 0 & 0 \\
E_j M & 0 & \omega_j M
\end{bmatrix}
\begin{bmatrix}
q \\
\dot{q} \\
q_v
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
F \\
0
\end{bmatrix}
\]
Frequency independent models

- Proved methodologies ADF (Lesieutre), GHM (Gola, …) , Prony series (abaqus)
  + Integrates into standard solvers
  + Time equivalent
    - High order for good material (solvers need to account for block structure)
A major computational challenge

- Sandwich oil pan
- 58766 DOFs
A major computational challenge

- NASTRAN 70.7 & SDT 5
- 58766 DOFs

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<th>SDT</th>
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<tr>
<td>F/B substitution (1 vect)</td>
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</table>

10 temperatures
1000 frequency

NASTRAN direct : 9 days
SDT Iterative : 612 s
Speedup : 1300

Still work to be done on accuracy/performance trade-off

More examples here
System vs. local

- System models (dynamic behaviour)
- FEM models (geometry/material knowledge)

Each objective requires different assumptions
System damping. 1 DOF example.

a) Bode
b) Nyquist
c) 1DOF model
d) Dynamic stiffness

dynamic stiffness equal at resonance

⇒
Response is the same

Different local models, same system response
Proportional damping model

- Modal damping is a good for system not for local
- Forced response along first mode

\[
\{q\} = \{\phi_j\} \cos(\omega_j t) \quad \text{Im}(K) \{\phi_j\} \cos(\omega_j t) \quad (M\phi_j) 2\zeta_j\omega_j \cos(\omega_j t)
\]

Modal looses localization of damping
**Systems and material level tests**

- Based on system level test, you get **test derived damping ratio for system model** they
  - are difficult/impossible to extrapolate to other system configurations
  - require matching of test/FEM modes
  - can rarely be translated in local damping information
- Based on **materials/components tests**, you get
  - damped FEM models, which are still difficult to solve
  - are only valid if almost all damping comes from well characterized parts
- Most damping models are arbitrary design parameters
Conclusion

- A lot of complex shapes are just not modes
- Some of them do come from damping
- Damping is becoming part of design
- We will see more of complex modes

www.sdtools.com/Publications.html
System damping model

Balmès IMAC 97: you can build equivalent system models in modal coordinates.
To obtain 1% modal damping:

- structural material damping leads to sparse matrix
- Viscous material damping leads to full matrix
Complex modes history

Just a few authors who talked about complex modes at IMAC

Balmes 94, Chung 87, Debao 87, Ewins 93, Gladwell 95, Hamidi 89, Ibrahim 83, 93 Imregun 91, 93, Inman 86, 95, Jun 84, Kirshenboin 87, Lallement 84, 87, Mitchell 90, 92, Montgomery 93, Niedbal 84, Ozguven 82, 86, Sas 92, Sestieri 93, Wei 87, Wicks 96, Zhang 84 …
Floor panel design

Floor panel (7998 nodes, 7813 elements)
+ free layers 2195 nodes, 1908 elements or
+ constrained layer 6595 n, 3816 elts)

NASTRAN element formulations

Objective: propose design steps
Validation addressed elsewhere
Design problems

- Basic design: selection of relevant materials
- Thickness optimization
- Treatment nature and position
  - A1 free layer 2.47 mm viscoelastic
  - B1 constrained layer: 50 mm visco, 0.3 mm steel
Material Selection

1. Select frequency range
2. Select temp range
3. Validate relevance on nomogram

Other considerations
Manufacturing, price, outgazing, aging, oil, ...

SM50e
PL3023
Temperature robustness validation

For a selected design, performance is judged by FRFs and Poles. Sensitivity to temperature must be evaluated by $B_1/T_a$ and $B_1/SM50e$. 

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- **Damping**
  - Material, Joint, ...
  - Damped models
  - Local/system model
Thickness optimization

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Local /

\[ K_{vi}(h_v, E_i) \approx \frac{h_v^0}{h_v} \frac{E_i(s, T, \sigma_0)}{E_0} K_{vi}(E_0) \]

\[ K_{ci}(h_c) \approx \frac{h_c^0}{h_c} K_{ci}(E_i) \]
A1 (Free layer) mass = 2 x B1 mass
B1-Ta very as efficient as A1 at 20°C
B1-SM very robust but not very good